Examples for Chapter 8.

EXAMPLE 7.1

A certain real random process has the exponential correlation function

$$R_x[l] = (0.5)^{|l|}$$

It is desired to compute the coefficients of the second order linear predictive filter and the corresponding prediction error variance.

Here the needed correlation terms are $R_x[0] = 1$, $R_x[1] = R_x[-1] = 0.5$, and $R_x[2] = R_x[-2] = 0.25$. The Normal equations have the form

$$\begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sigma_\varepsilon^2 \\ 0 \\ 0 \end{bmatrix}$$

This can be broken up into a set of equations for the unknown filter coefficients

$$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.25 \end{bmatrix}$$

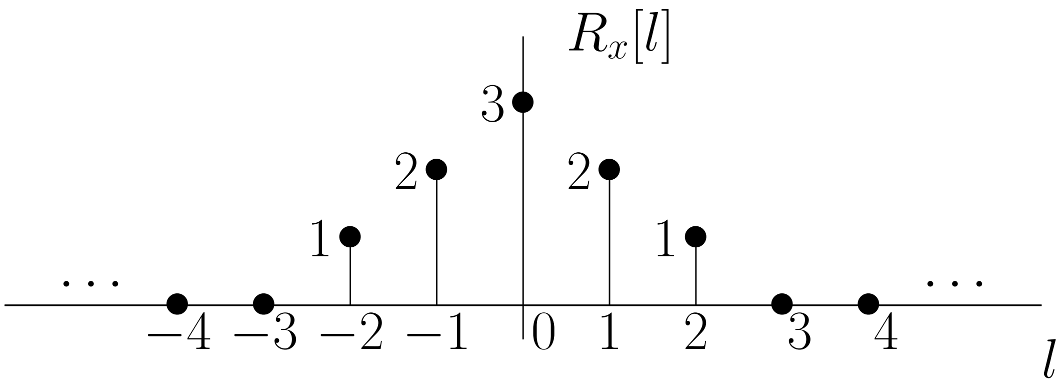
which has the solution $a_1 = -0.5$ and $a_2 = 0$, and a separate equation for the prediction error variance which can then be solved as

$$\begin{aligned}\sigma_\varepsilon^2 &= R_x[0] + a_1 R_x[1] + a_2 R_x[2] \\ &= 1 + (-0.5)(0.5) + (0)(0.25) = 0.75\end{aligned}$$

The fact that a_2 is zero for this particular correlation function is no coincidence and will be clearly understood when we investigate linear prediction further. $\square$

EXAMPLE 8.1

Consider the linear prediction of a real random process with the correlation function shown below:



It is desired to find the second order linear prediction parameters for the random process $x[n]$ and the values of the reflection coefficients. The first three values of the correlation function are

$$R_x[0] = 3\,; \quad R_x[1] = R_x[-1] = 2\,; \quad R_x[2] = R_x[-2] = 1$$

Therefore the Normal equations for this problem are

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sigma_2^2 \\ 0 \\ 0 \end{bmatrix}$$

The equations are solved by applying the Levinson recursion.

Begin with $\mathbf{r}_0 = R_x[1] = 2$, $\boldsymbol{a}_0 = 1$, and $\sigma_0^2 = R_x[0] = 3$ and proceed through the first stage of the recursion (for $p = 1$):

$$\gamma_1 = \frac{\mathbf{r}_0 \boldsymbol{a}_0}{\sigma_0^2} = \frac{2 \cdot 1}{3} = \tfrac{2}{3}$$

$$\boldsymbol{a}_1 = \begin{bmatrix} \boldsymbol{a}_0 \\ -- \\ 0 \end{bmatrix} - \gamma_1 \begin{bmatrix} 0 \\ -- \\ \tilde{\boldsymbol{a}}_0^* \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \tfrac{2}{3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\tfrac{2}{3} \end{bmatrix}$$

$$\sigma_1^2 = (1 - |\gamma_1|^2)\sigma_0^2 = (1 - (\tfrac{2}{3})^2) \cdot 3 = \tfrac{5}{3}$$

This determines the first order parameters including the reflection coefficient γ_1. The next stage of recursion proceeds in a similar fashion (for $p = 2$):

$$\gamma_2 = \frac{\mathbf{r}_1^{*T} \tilde{\boldsymbol{a}}_1}{\sigma_1^2} = \frac{1}{\frac{5}{3}} \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} -\tfrac{2}{3} \\ 1 \end{bmatrix} = -\tfrac{1}{5} = -0.2$$

$$\boldsymbol{a}_2 = \begin{bmatrix} \boldsymbol{a}_1 \\ -- \\ 0 \end{bmatrix} - \gamma_2 \begin{bmatrix} 0 \\ -- \\ \tilde{\boldsymbol{a}}_1^* \end{bmatrix} = \begin{bmatrix} 1 \\ -\tfrac{2}{3} \\ 0 \end{bmatrix} - (-\tfrac{1}{5}) \begin{bmatrix} 0 \\ -\tfrac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\tfrac{4}{5} \\ \tfrac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

$$\sigma_2^2 = (1 - |\gamma_2|^2)\sigma_1^2 = (1 - | - \tfrac{1}{5}|^2) \cdot \tfrac{5}{3} = \tfrac{8}{5} = 1.6$$

This completes the solution. $\square$

EXAMPLE 8.2

It is desired to find the reflection coefficients in the lattice representation of the prediction error filter with coefficients

$$\boldsymbol{a}_2 = \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

Begin by observing that

$$\gamma_2 = -a_2^{(2)} = -0.2$$

Then by using the reverse order recursion formula:

$$\begin{bmatrix} \boldsymbol{a}_1 \\ \\ -- \\ 0 \end{bmatrix} = \frac{1}{1-|\gamma_2|^2}\left[\boldsymbol{a}_2 + \gamma_2 \tilde{\boldsymbol{a}}_2^*\right]$$

$$= \frac{1}{1-|-0.2|^2}\left[\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} + (-0.2)\begin{bmatrix} 0.2 \\ -0.8 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 1 \\ -0.666\ldots \\ 0 \end{bmatrix}$$

Thus

$$\gamma_1 = +0.666\ldots$$

These correspond to the refection coefficients found in Example 8.1. Obviously if the original filter order had been higher, the process could have been continued until all the reflection coefficients were calculated.

The prediction error variance of the first order filter in this example is computed as

$$\sigma_1^2 = \frac{\sigma_2^2}{1 - |\gamma_2|^2} = \frac{1.6}{1 - |-0.2|^2} = 1.666\ldots$$

which again agrees with the result of Example 8.1. $\square$

EXAMPLE 8.3

It is desired to compute the reflection coefficients and prediction error variances for the prediction error filter of Example 8.1 by the Schur algorithm. The algorithm is initialized with the real correlation function

$$\mathbf{G}_0 = \begin{bmatrix} \times & R_x^*[1] & R_x^*[2] \\ R_x[0] & R_x^*[1] & R_x^*[2] \end{bmatrix} = \begin{bmatrix} \times & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

In the first step ($p = 1$) γ_1, $\mathbf{G}_1$ and σ_1^2 are computed:

$$\gamma_1 = \frac{g_0[1]}{g_0^b[0]} = \frac{2}{3} = \tfrac{2}{3}$$

$$\begin{aligned} \mathbf{G}_1 &= \mathbf{H}_1^* \mathbf{G}_0^{(shift)} \\ &= \begin{bmatrix} 1 & -\frac{2}{3} \\ -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} \times & 2 & 1 \\ \times & 3 & 2 \end{bmatrix} = \begin{bmatrix} \times & 0 & -\frac{1}{3} \\ \times & \frac{5}{3} & \frac{4}{3} \end{bmatrix} \end{aligned}$$

$$\sigma_1^2 = g_1^b[1] = \tfrac{5}{3}$$

In the second step ($p = 2$) the remaining terms are computed similarly:

$$\gamma_2 = \frac{g_1[2]}{g_1^b[1]} = \frac{-1/3}{5/3} = -\tfrac{1}{5}$$

$$\begin{aligned} \mathbf{G}_2 &= \mathbf{H}_2^* \mathbf{G}_1^{(shift)} \\ &= \begin{bmatrix} 1 & \frac{1}{5} \\ \frac{1}{5} & 1 \end{bmatrix} \begin{bmatrix} \times & 0 & -\frac{1}{3} \\ \times & \times & \frac{5}{3} \end{bmatrix} = \begin{bmatrix} \times & \times & 0 \\ \times & \times & \frac{8}{5} \end{bmatrix} \end{aligned}$$

$$\sigma_2^2 = g_2^b[2] = \tfrac{8}{5}$$

The results are seen to agree with those of Example 8.1. □

EXAMPLE 8.4

It is desired to compute the second order linear prediction parameters for the random process whose correlation function is given in Example 8.1. (Recall that $R_x[0] = 3$, $R_x[1] = 2$, and $R_x[2] = 1$.) The computation is carried out using the complex split Levinson algorithm. The algorithm is initialized with

$$\tau_0 = R_x[0] = 3\,; \quad \boldsymbol{s}_0 = 2\,; \quad \boldsymbol{s}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In the first step of the recursion ($p = 1$) τ_1, β_1, and $\boldsymbol{s}_2$ are computed.

$$\begin{aligned} \tau_1 &= \sum_{k=0}^{0} \left(R_x[k] + R_x[1-k]\right) s_k^{(1)} \\ &= \left(R_x[0] + R_x[1]\right) s_0^{(1)} = (3+2)\cdot 1 = 5 \end{aligned}$$

$$\beta_1 = \frac{\tau_0}{\tau_1} = \frac{3}{5} = \tfrac{3}{5}$$

$$
\begin{aligned}
\boldsymbol{s}_2 &= \beta_1 \left(\begin{bmatrix} \boldsymbol{s}_1 \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ \boldsymbol{s}_1 \end{bmatrix} \right) - \begin{bmatrix} 0 \\ \boldsymbol{s}_0 \\ 0 \end{bmatrix} \\
&= \tfrac{3}{5} \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) - \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ \times \end{bmatrix}
\end{aligned}
$$

(The symbol $\times$ here indicates that this element does not have to be computed due to the symmetry.)

In the next step of the recursion ($p = 2$) τ_2, β_2, and $\boldsymbol{s}_3$ are computed.

$$
\begin{aligned}
\tau_2 &= R_x[1] s_1^{(2)} + \sum_{k=0}^{0} \left(R_x[k] + R_x[2-k] \right) s_k^{(1)} \\
&= R_x[1] s_1^{(2)} + \left(R_x[0] + R_x[2] \right) s_0^{(2)} \\
&= 2 \left(-\tfrac{4}{5} \right) + (3+1) \tfrac{3}{5} = \tfrac{4}{5}
\end{aligned}
$$

$$\beta_2 = \frac{\tau_1}{\tau_2} = \frac{5}{\frac{4}{5}} = \tfrac{25}{4}$$

$$\begin{aligned}
\boldsymbol{s}_3 \;&=\; \beta_2 \left(\begin{bmatrix} \boldsymbol{s}_2 \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ \boldsymbol{s}_2 \end{bmatrix} \right) - \begin{bmatrix} 0 \\ \boldsymbol{s}_1 \\ 0 \end{bmatrix} \\
&= \tfrac{25}{4} \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ \frac{3}{5} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{3}{5} \\ -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{15}{4} \\ -\frac{9}{4} \\ \times \\ \times \end{bmatrix}
\end{aligned}$$

The filter coefficients are then computed from:

$$\begin{bmatrix} \boldsymbol{a}_2 \\ \\ 0 \end{bmatrix} = C_3' \boldsymbol{s}_3 - K_2' \begin{bmatrix} 0 \\ \\ \boldsymbol{s}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ \boldsymbol{a}_2 \end{bmatrix}$$

where

$$C_3' = \frac{1}{\operatorname{top}(\boldsymbol{s}_3)} = \tfrac{4}{15}$$

and

$$K_2' = \frac{\boldsymbol{\Sigma}(\boldsymbol{s}_3)}{\boldsymbol{\Sigma}(\boldsymbol{s}_2)} C_3' = \frac{3}{\frac{2}{5}} \cdot \tfrac{4}{15} = 2$$

Writing this in component form we have

$$\begin{bmatrix} 1 \\ a_1^{(2)} \\ a_2^{(2)} \\ 0 \end{bmatrix} = \tfrac{4}{15} \begin{bmatrix} \frac{15}{4} \\ -\frac{9}{4} \\ -\frac{9}{4} \\ \frac{15}{4} \end{bmatrix} - 2 \begin{bmatrix} 0 \\ \frac{3}{5} \\ -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ a_1^{(2)} \\ a_2^{(2)} \end{bmatrix}$$

The components are then computed one at a time.

$$a_1^{(2)} = \tfrac{4}{15}\left(-\tfrac{9}{4}\right) - 2\left(\tfrac{3}{5}\right) + 1 = -\tfrac{4}{5}$$

$$\begin{aligned} a_2^{(2)} &= \tfrac{4}{15}\left(-\tfrac{9}{4}\right) - 2\left(-\tfrac{4}{5}\right) + a_1^{(2)} \\ &= -\tfrac{3}{5} + \tfrac{8}{5} - \tfrac{4}{5} = \tfrac{1}{5} \end{aligned}$$

The coefficient vector is therefore

$$\boldsymbol{a}_2 = \begin{bmatrix} 1 \\ -\frac{4}{5} \\ \frac{1}{5} \end{bmatrix}$$

Finally σ_2^2 is computed from:

$$\sigma_2^2 = K_2' \tau_2 = 2\left(\tfrac{4}{5}\right) = \tfrac{8}{5}$$

These are seen to check with the results of Example 8.1. $\square$

EXAMPLE 8.5

It is desired to compute the reflection coefficients and prediction error variances up to second order for the random process in Example 8.1 with

$$R_x[0] = 3 \qquad R_x[1] = 2 \qquad R_x[2] = 1$$

This is done using the complex split Schur algorithm.

The algorithm is initialized with

$$\boldsymbol{v}_0^{(s)} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}; \qquad \boldsymbol{v}_1 = \begin{bmatrix} \boldsymbol{v}_1^{(s)} \\ \\ v_1[2] \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}; \qquad C_1 = 1\,; \qquad \xi_1 = \xi_0 = 2$$

The first step ($p = 1$) consists of the following computations:

$$\beta_1 = \frac{\mathrm{top}(\boldsymbol{v}_0)}{\mathrm{top}(\boldsymbol{v}_1)} = \frac{3}{5} = \tfrac{3}{5}$$

$$\xi_2 = 2\beta_1\xi_1 - \xi_0 = \tfrac{6}{5} \cdot 2 - 2 = \tfrac{2}{5}$$

$$C_2 = \beta_1 C_1 = \tfrac{3}{5} \cdot 1 = \tfrac{3}{5}$$

$$K_1 = C_2 \frac{\xi_1}{\xi_2} = \tfrac{3}{5} \cdot \frac{2}{\tfrac{2}{5}} = 3$$

$$\gamma_1 = \frac{1}{K_1}(K_1 - C_1) = \frac{1}{3}(3-1) = \tfrac{2}{3}$$

$$\boldsymbol{\sigma}_1^2 = \frac{\mathrm{top}(\boldsymbol{v}_1)}{K_1} = \tfrac{5}{3}$$

and finally

$$\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ v_2[2] \end{bmatrix} = \beta_1 \left(\begin{bmatrix} \boldsymbol{v}_1^{(s)} \\ \\ v_1[2] \end{bmatrix} + \begin{bmatrix} 0 \\ \\ \boldsymbol{v}_1^{(s)} \end{bmatrix} \right) - \boldsymbol{v}_0^{(s)}$$

$$= \tfrac{3}{5} \left(\begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} \right) - \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ \tfrac{4}{5} \end{bmatrix}$$

This produces the first order parameters γ_1, $\boldsymbol{\sigma}_1^2$ and the vector $\boldsymbol{v}_2$ needed for the next time through the recursion.

At the next and final step ($p = 2$) the process is repeated:

$$\beta_2 = \frac{\mathrm{top}(\boldsymbol{v}_1)}{\mathrm{top}(\boldsymbol{v}_2)} = \frac{5}{\frac{4}{5}} = \tfrac{25}{4}$$

$$\xi_3 = 2\beta_2\xi_2 - \xi_1 = \tfrac{25}{2} \cdot \tfrac{2}{5} - 2 = 3$$

$$C_3 = \beta_2 C_2 = \tfrac{25}{4} \cdot \tfrac{3}{5} = \tfrac{15}{4}$$

$$K_2 = C_3 \frac{\xi_2}{\xi_3} = \tfrac{15}{4} \frac{\frac{2}{5}}{3} = \tfrac{1}{2}$$

$$\gamma_2 = \frac{1}{K_2}(K_2 - C_2) = 2\left(\tfrac{1}{2} - \tfrac{3}{5}\right) = -\tfrac{1}{5}$$

$$\sigma_2^2 = \frac{1}{K_2}\mathrm{top}(\boldsymbol{v}_2) = 2 \cdot \tfrac{4}{5} = \tfrac{8}{5}$$

This completes the computation of the parameters γ_p and σ_p^2 up to order 2. The computation of $\boldsymbol{s}_3$ is skipped since all of the desired parameters have been found. The results check with those obtained in the earlier examples. $\square$

Ex.8.5(3)